Visvesvaraya National Institute of Technology, Nagpur

Department of Mathematics

Assignment-3 **Topology (MAL522)**

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1. Let d be a metric on a set X Show that $d_0(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$, where $x, y \in X$ is also a metric on X

2. Let (X_1, d_1) and (X_2, d_2) represent metric spaces. Show that $(X_1 \times X_2, d)$ where

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

   for $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$, is a metric space.

3. Let $B(x_0, r)$ be a ball in a metric space. Show that for every point $y \in B(x_0, r)$, there exists a ball $B(y, r^{'})$ such that $B(y, r^{'}) \subset B(x_0, r)$

4. Prove that if B_1 and B_2 are balls with the same center (in metric space), then one of them is a subset of the other.

5. Is it possible for a convergent sequence to have two different limits in a metric space?

6. A sequence of points $z_n = (x_n, y_n)$ of the space $Z = X \times Y$ is convergent to a point $z = (x, y)$ if and only if $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$. $(X, Y, Z$ are metric spaces)

7. Prove that every metric space is Hausdorff.

8. Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

9. Let (X, d) be a metric space. Consider the family T of all d-open subsets of X, i.e., subsets which are open in the sense of metric d. Show that T is a topology on (X, d).

10. Let $G \cup H$ be a disconnection of A. Show that $A \cap G$ and $A \cap H$ are separated sets.

11. Prove that a set A is connected if and only if A is not the union of two non-empty separated sets.

12. If A and B are connected sets which are not separated, then $A \cup B$ is connected.

13. Show that any set X, consisting of two or more points, with discrete topology is disconnected.

14. Let X be a topological space. Show that the following conditions are equivalent: (i) X is disconnected, (ii) There exists a non-empty proper subset of X which is both open and closed.

15. Show that if (X, T) is any topological space then the following statements are equivalent: (1) X is disconnected. (2) It cannot be expressed as the union of nonempty, disjoint closed subsets.

16. Let A be a connected subset of (X, T) and $A \subset B \subset \overline{A}$. Show that B is connected.

17. Show that the only connected subsets of $\mathbb{R}$ with the usual topology, having more than one point, are $\mathbb{R}$ and the intervals (open, closed, half-open).

18. Show that if (X, T) is a path-connected space, then X is connected.

19. Show that every component E is closed.

20. The components of X form a partition of X. Every connected subset of X is contained in some component.

21. Prove that the product of connected spaces is connected (in the Product topology).

22. A topological space X is said to be totally disconnected if for each pair of points $p, q \in X$ there exists a disconnection $G \cup H$ of X with $p \in G$ and $q \in H$. Show that the real line $\mathbb{R}$ with the topology T generated by the open closed intervals $(a, b]$ is totally disconnected.

23. Show that every finite subset of a topological space (X, T) is compact.

24. Let $\mathbb{N}$ represent the set of natural numbers with topology T defined as follows: a subset $U \subset N$ is open if it contains all, but finitely many elements of $\mathbb{N}$. Show that $(\mathbb{N}, T)$ is a compact space.

25. Show that any infinite subset A of a discrete topological space X is not compact.

26. Show that a topological space (X, T) with the cofinite topology is compact.

27. Is every subset of a compact space is compact?

28. A topological space X is compact if and only if every class $\{F_i\}$ of closed subsets of X which satisfies the finite intersection property has, itself, a non-empty intersection.

29. Let A and B be disjoint compact subsets of a Hausdorff space X. Then there exist disjoint open sets G and H such that $A \subset G$ and $B \subset H$.

30. Every compact space is locally compact.

31. Let $G = \{G_i\}$ be an open cover of a sequentially compact set A. Then G has a (positive) Lebesgue number.

Sequence Lemma: If X is first countable and $A \subset X$, then $x \in \overline{A}$ if and only if there is a sequence (x_n) contained in A which converges to x.

If $x \in \overline{A}$, pick a countable decreasing nhd base $\{U_n\}$ at x. Then choosing $x_n \in U_n \cap A$ provides the required sequence (note that since $x \in \overline{A}$ the sets $U_n \cap A$ are nonempty).