Department of Mathematics, Visvesvaraya National Institute of Technology Nagpur
Mathematics I (MAL 101)

Problems Sheet 1 (Matrices) **(Common for all Sections)**

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1. Find the rank of following matrices:
(i) $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix}$

2. For what values of a, b the rank of matrix $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & a & b \end{bmatrix}$ is 2.

3. Give a system of linear equations having
(i) $(1, 0)$ as the only solution. (ii) $(1, 0)$ and $(0, 1)$ as solutions.

4. (i) Let A be an $n \times n$ matrix. If the system $A^2\mathbf{x} = 0$ has non-trivial solution then show that $A\mathbf{x} = 0$ also has a non-trivial solution.
(ii) Suppose $\mathbf{x}_1, \mathbf{x}_2$ are two solutions of $A\mathbf{x} = 0$. Then $k_1\mathbf{x}_1 + k_2\mathbf{x}_2$ is also a solution of $A\mathbf{x} = 0$ for any $k_1, k_2 \in \mathbb{R}$.
(iii) If u and v are two solutions of $A\mathbf{x} = b$ then $u - v$ is a solution of the system $A\mathbf{x} = 0$.

5. (i) Find the parabola $y = A + Bx + Cx^2$ that passes through three points $(1, 1)$, $(2, -1)$ and $(3, 1)$.
(ii) Let $P(x) = a_0 + a_1x + a_2x^2$. Choose a_i such that $P(1) = b_1, P(2) = b_2, P(3) = b_3$. Is this choice unique?

6. (i) Find the condition on a, b, c so that the linear system $x+2y-3z = a, 2x+6y-11z = b, x-2y+7z = c$ is consistent.
(ii) Determine the values of a and b, if they exists, such that the system $x+y+z = 6$; $x+2y+3z = 10$; $x + 2y + az = b$ has (a) no solution (b) a unique solution (c) infinitely many solutions.

7. (i) Find all possible solutions of $x + y - z = 1$.
(ii) Is the system $x + y - z = 1$; $x + y + z = -1$ consistent? If so, find its solution(s)?

8. (i) For what values of p, the system $x + y + z = 1$; $x + 2y + 4z = p$; $x + 4y + 10z = p^2$ has a solution and solve it completely in each case.
(ii) Find the value of p, for which the system $A\mathbf{x} = b$,
where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & p+1 & p \\ 0 & 0 & p-2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 & -1 & 3p \end{bmatrix}^t$ has a solution and solve it.
(iii) If $A = \begin{bmatrix} 6 & 4 & -1 \\ -4 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, prove that the system $A\mathbf{x} = \lambda\mathbf{x}$ is consistent system if and only if $\lambda = 1$ or 2 and find solution in each case.
(iv) Show that the system $x_1-2x_2+x_3+4x_4 = 1,\ x_1+x_2-x_3+2x_4 = 2,\ x_1+\alpha x_2-5x_3-2x_4 = 3$ has no solution if and only if $\alpha = 7$. If $\alpha \neq 7$, find the general solution.

9. Let $A = (a_{ij})_{n\times n}$, $a_{ij} = 1$ for $1 \leq i, j \leq n$. Solve the system $A\mathbf{x} = b$ with
(a) $b = [0, 0, ...0]^T$, (b) $b = [1, 1, ...1]^T$.

10. If $X_1, X_2, ..., X_n$ are solutions of $AX = b$.
Show that $\alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_n X_n$ is also a solution of $AX = b$ if $\alpha_1 + \alpha_2 + ... + \alpha_n = 1$.

11. Find the inverse of the following matrices using Gauss-Jordan method.
(i) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -1 & 3 \\ -1 & 3 & -2 \\ 2 & 4 & 1 \end{bmatrix}$.

12. Check whether the following sets are linearly independent or not.
(i) $\{(1,3,2),(2,1,3),(3,2,1)\}$ (ii) $\{(1,2,-1),(3,6,-3),(3,9,3),(2,5,0)\}$
(iii) $\{(1,1,1),(1,2,3),(1,5,8)\}$ (iv) $\{(1,0,0,0),(0,1,0,0),(1,2,0,1),(0,0,0,1)\}$

13. Show that the set of vectors $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{u}-2\mathbf{v}+\mathbf{w}\}$ are linearly independent if the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent.

14. Explain the following briefly:
(i) If $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is a linearly independent set. Is the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly independent?
(ii) If $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is a linearly dependent set. Is the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ linearly dependent?
(iii) The singleton set $\{\mathbf{x}\}$, where $\mathbf{x}$ is a non-zero vector, is linearly independent.
(iv) Can the row vectors of a 20×15 matrix be linearly independent?

15. (i) Is $(4,5,5)$ a linear combination of $(1,0,0),(1,1,0)$ and $(1,1,1)$?
(ii) Is $(4,5,5)$ a linear combination of $(1,2,3),(-1,1,4)$ and $(3,3,2)$?
(iii) Show that $(4,5,5)$ is not a linear combination of the vectors $(1,2,1)$ and $(1,1,0)$?

16. Find the inverse of the following matrices by using the Clayey-Hamilton Theorem
(i) $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 1 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -2 & -1 \\ -2 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

17. Find the spectrum and eigenvectors of the following matrices:
(i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{bmatrix}$ (vi) $\begin{bmatrix} -10 & 10 & -15 \\ 10 & 5 & -30 \\ -5 & -10 & 0 \end{bmatrix}$

18. Find the eigen pairs of the matrices (i) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

19. Let A be an $n \times n$ matrix and $m \geq 1$ be an integer. Show that
(i) If λ is an eigenvalue of A then λ^m is an eigenvalue of A^m
(ii) If λ is an eigen value of A then $\lambda + \lambda^2$ is an eigenvalue of $A + A^2$
(iii) If A is ivertible matrix then the eigenvalues of A^{-1} are reciprocal of the eigenvalues of A, with the same eigenvectors
(iv) If α is a scalar then find the eigenvalues of $A - \alpha I$ in terms of eigenvalues of A. Further show that A and $A - \alpha I$ have the same eigenvectors.

20. Let A be an $n \times n$ matrix. Show that:
(i) If A is Hermitian then all eigenvalues of A are real.
(ii) If A is skew-Hermitian then all eigenvalues of A are imaginary.
(iii) If A is real and symmetric then all eigenvalues of A are real.
(iv) If A is unitary matrix then all the eigenvalues of A are of magnitude 1.

21. Let A be a $n \times n$ matrix. Show that A^t and A have the same eigenvalues. Do they have the same eigenvectors?

22. Let A be an $n \times n$ matrix. Show that:
(i) If $A$is idempotent $(A^2 = A)$ then eigenvalues of A are either 0 or 1.
(ii) If A is nilpotent $(A^m = 0$ for some $m \geq 1)$ then all eigenvalues of A are 0.

23. If A and B are $n \times n$ matrices with A non-singular, then prove that $A^{-1}B$ and BA^{-1} have the same set of eigenvalues.

24. Let $(\lambda_1, \mathbf{u})$ be an eigen pair for a matrix A and let $(\lambda_2, \mathbf{u})$ be an eigen pair for another matrix B.
(i) Then prove that $(\lambda_1 + \lambda_2, \mathbf{u})$ is an eigen-pair for the matrix $A + B$.
(ii) Give an example to show that if λ_1, λ_2 are respectively the eigenvalues of A and B, then $\lambda_1 + \lambda_2$ need not be an eigenvalue of $A + B$.

25. (i) Let A be 3×3 matrix having eigenvalues $1, 2, -1$. Find the trace of the matrix $B = A - A^{-1} + A^2$.
(ii) Let A be 4×4 matrix having eigenvalues $1, 2, 3, 4$. Find the determinant of the matrix $B = 2A + A^{-1} - I$.

26. Let A be a matrix and S be an invertible matrix. Show that eigenvalues of A and $S^{-1}AS$ are same.

27. If A and B are similar matrices then show that
(i) A^{-1} and B^{-1} are similar (ii) A^m and B^m are similar, $m \in \mathbb{N}$ (iii) $|A| = |B|$.

28. Determine whether following are similar matrices:
(i) $A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

29. Check whether the eigenvectors of the following matrices are linearly independent or not, if so find the matrix P such that $P^{-1}AP$ is diagonal matrix:
(a) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

30. Find the matrix A whose eigen pairs are given as follows:
(i) $(2, (-2, 1, 0)^t)$, $(2, (-1, 0, 1)^t)$, $(4, (1, 0, 1)^t)$
(ii) $(1, (1, 2, 1)^t)$, $(2, (2, 3, 4)^t)$, $(3, (1, 4, 9)^t)$
(iii) $(0, (-1, 1, 0)^t)$, $(-1, (1, 0, -1)^t)$, $(1, (1, 1, 1)^t)$

31. Let A be an $n \times n$ complex matrix and let $\mathbf{x}, \mathbf{y}$ be eigenvectors of A corresponding to distinct eigenvalues of A. Show that $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent.

32. Let A be a square matrix of order n. Prove that if $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ are the linearly independent eigenvectors of A, then A is diagonalisable.

33. (i) Show that the matrices $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & -2 \\ 0 & 3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ are diagonalisable.
(ii) Show that the matrices $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & i \\ i & 0 \end{bmatrix}$ are not diagonalisable.

34. Let A be a non-zero square matrix such that $A^2 = 0$. Show that A cannot be diagonalized.

35. Reduce the following quadratic equations to canonical form:
(i) $3x^2 + 2y^2 - 2xy = 6$, (ii) $2x^2 - 73xy + 23y^2 - 50 = 0$, (iii) $xy = 1$,
(iv) $4y^2 + 3xy = 1$ (v) $x^2 + y^2 - 10xy = 4$.