Visvesvaraya National Institute of Technology, Nagpur
Department of Mathematics
Assignment-Integral Calculus: MAL101

1. If $f(1) = 2$, f' is continuous and $\int_1^4 f'(x)dx = 17$. What is the value of $f(4)$?.

1. Length of the curves

   (a) Determine the perimeter of one loop of the curve $6ay^2 = x(x - 2a)^2$ [Ans. $\frac{8a}{\sqrt{3}}$]

   (b) Calculate the distance traveled by the particle $P(x, y)$ after 4 minutes, if the position at any time is given by $x = \frac{t^2}{2}$, $y = \frac{1}{3}(2t + 1)^{3/2}$. [Ans. 12]

   (c) Find the perimeter of the curve $r = a(\cos\theta + \sin\theta)$ [Ans. $\sqrt{2}\pi a$]

   (d) Determine the perimeter of the curve $r = a\sin^3(\frac{\theta}{3})$ [Ans. $\frac{3\pi a}{2}$]

   (e) Find the length of the arc of the parabola $r = \frac{2a}{(1+\cos\theta)}$ cut-off by its latus rectum. [Ans. $(\sqrt{2} + \ln(1 + \sqrt{2}))2a$]

   (f) Find the length of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$. [Ans. $6a$]

2. Area the curves

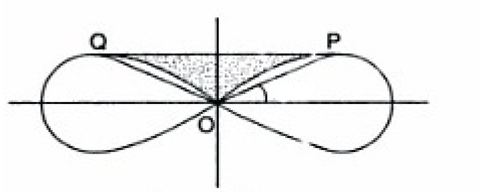
   (a) Determine the area between the cubic $y = x^3$ and the parabola $y = 4x^2$. [Ans. $\frac{64}{3}$]

   (b) Calculate the area between the curve $y^2(a + x) = (a - x)^3$ and its asymptotes. [Ans. $3\pi a^2$]

   (c) Find the whole area bounded by the four infinite branches of the tractrix: $x = a\cos t + \frac{1}{2}a\ln\tan^2\frac{t}{2}$, $y = a\sin t$. [Ans. πa^2]

   (d) Find the whole area of the curves (i) $r = a\cos n\theta$ (ii) $r = a\sin n\theta$ (iii) $r = a\cos 3\theta + b\sin 3\theta$. [Ans. (i) $\frac{\pi a^2}{4n}$ (ii) $\frac{\pi a^2}{4n}$, (iii) $\frac{\pi(a^2+b^2)}{4}$]

   (e) Let PQ be the common tangent to the two loops of the lemniscate $r^2 = a^2\cos 2\theta$ with pole O. Find the area bounded by the line PQ and the arcs OP and OQ of the curve.



   [Ans. $\frac{a^2}{8}(3\sqrt{3} - 4)$]

   (f) Compute the area bounded by the x-axis and an arc of the cycloid $x = a(t - \sin t)$,$y = a(1 - \cos t)$. [Ans. $3\pi a^2$]

3. Volume and Surfaces of Solid of Revolution

   (a) Find the volume of the solid generated by the revolution of an arc of the catenary $y = c\cosh(x/c)$ about the x-axis. [Ans. $\frac{\pi c^2}{2}(x + \frac{c}{2}\sinh(2x/c))$]

   (b) Determine the volume of solid generated by revolving the plane area bounded by $y^2 = 4x$ and $x = 4$ about the line $x = 4$. [Ans. $\frac{1024}{15}\pi$]

   (c) Find the volume of the solid generated by revolving the smaller area bounded by the circle $x^2 + y^2 = 2$ and semicubical parabola $y^3 = x^2$ about the x-axis [Ans. $\frac{52}{21}\pi$]

   (d) Determine the volume of solid of revolution generated by revolving the curve whose parametric equation are $x = 2t + 3$, $y = 4t^2 - 9$ about the x-axis for $t_1 = -\frac{3}{2}$, $t_2 = \frac{3}{2}$. [Ans. 1296π]

   (e) The arc of the cardioid $r = a(1 + \cos\theta)$ included between $\theta = -\pi/2$ and $\theta = \pi/2$ is rotated about the line $\theta = \pi/2$. Find the volume of the solid of revolution. [Ans. $\frac{\pi a^3}{4}(16 + 5\pi)$]

   (f) Find the volume of the solid generated by the revolution of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{\frac{-x}{a}})$ about the x-axis between $x = 0$ and $x = \frac{\pi}{2}$. [Ans. $\frac{\pi a^3}{8}(e^{2b/a} - e^{-2b/a}) + \frac{\pi a^2 b}{2}$]

(g) Find the volume of the solid obtained by rotating about the y-axis, the region bounded by $y = 2x^2 - x^3$ and $y = 0$. [Ans. $\frac{16\pi}{5}$]

(h) Find the volume of the solid obtained by rotating about the x-axis, the region bounded by $x = 1 + y^2$, $x = 0$,$y = 1$, $y = 2$. [Ans. $\frac{21\pi}{2}$]

(i) Determine the surface of a paraboloid generated by revolution about the x-axis of an arc of the parabola $y^2 = 2px$, which corresponds to the variation of x from $x = 0$ to $x = a$. [Ans. $\frac{2\pi\sqrt{p}}{3}[(2a+p)^{3/2} - p^{3/2}]$]

(j) The astroid $x = a\sin^3 t$,$y = a\cos^3 t$ is revolved about the x-axis.Find the surface of the solid of revolution. [Ans. $\frac{12\pi a^2}{5}$]

(k) The arc of the parabola $y = x^2$ from $(1,1)$ to $(2,4)$ is rotated about the y-axis.Find the area of the resulting surface. [Ans. $\frac{\pi}{6}[17\sqrt{17} - 5\sqrt{5}]$]

(l) Find the volume of the solid obtained by revolving the cissoid $y^2(2a - x) = x^3$ about its asymptote. [Ans. $2\pi^2 a^3$]

(m) Find the volume of the solid obtained by revolving the area bounded by the curve $y^2 = x$ and the line $y = 4$ about the line $x = 2$. [Ans. $\frac{128\pi}{3}$]

(n) A circular arc revolves about its chord. Find the area of the surface generated, when 2α is the angle subtended by the arc at the center. [Ans. $4\pi a^2(\sin\alpha - \alpha\cos\alpha)$]

4. Beta and Gamma functions.

(a) Evaluate $\int_0^2 (8 - x^3)^{\frac{-1}{3}} dx$.

(b) Evaluate $\int_0^1 \frac{x^{(m-1)}(1-x)^{(n-1)}}{(a+bx)^{(m+n)}} dx$.

(c) Show that $\int_0^1 \frac{1}{\sqrt{(1-x^n)}} dx = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{n})}{n\Gamma(\frac{1}{n}+\frac{1}{2})}$.

(d) Show that $\int_0^{\frac{\pi}{2}} \sqrt{(\sin\theta)} d\theta \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(\sin\theta)}} d\theta = \pi$.

(e) Evaluate $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}} dx$.

(f) Show that $\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q}$, where p and q are positive.

5. Differentiation under integral sign.

(a) Evaluate $\int_0^\pi \frac{\ln(1+\sin\alpha\cos x)}{\cos x} dx$.

(b) Evaluate $\int_0^\infty e^{-(x^2+\frac{a^2}{x^2})} dx$.

(c) Evaluate $\int_0^\infty \frac{\arctan(ax)}{x(1+x^2)} dx$.

(d) Evaluate $\int_0^{\frac{\pi}{2}} (\frac{1+y\sin^2 x}{\sin^2 x}) dx$.

(e) Using the result $\int_0^\infty e^{(-x^2)} dx = \frac{\sqrt{\pi}}{2}$, evaluate $\int_0^\infty e^{(-x^2)}\cos(\alpha x) dx$.

Hence deduce that $\int_0^\infty x e^{(-x^2)}\sin(\alpha x) dx = \frac{\sqrt{\pi}}{4}\alpha e^{\frac{(-\alpha^2)}{4}}$