VISVESVARAYA NATIONAL INSTITUTE OF TECHNOLOGY, NAGPUR
DEPARTMENT OF MATHEMATICS
Assignment on Fourier Integrals & Fourier Transforms **MAL-201, B.Tech., III-Semester(2014).**

Fourier Integrals:

1. Find Fourier integral representation of the following functions:
(i) $f(x)=\begin{cases}0, & x<0\\ x, & 0\le x<2\\ 0, & x\ge 2\end{cases}$ (ii) $f(x)=\begin{cases}0, & x<0\\ 1, & 0\le x<1\\ 0, & x\ge 1\end{cases}$ (iii) $f(x)=\begin{cases}0, & x<0\\ e^{-x}, & x\ge 0\end{cases}$
(iv) $f(x)=\begin{cases}e^{x}, & |x|<2\\ 0, & |x|\ge 2\end{cases}$ (v) $f(x)=\begin{cases}e^{-|x|}, & |x|<1\\ 0, & \text{Otherwise}\end{cases}$ (vi) $f(x)=\begin{cases}\sin x, & -2\le x\le 0\\ \cos x, & 0<x\le 2\\ 0, & \text{Otherwise}\end{cases}$

2. Find Fourier Cosine integral of the following functions:
(i) $f(x)=\begin{cases}x^2, & 0\le x\le 5\\ 0, & x>5\end{cases}$ (ii) $f(x)=\begin{cases}\sin x, & 0\le x\le \pi\\ 0, & x>\pi\end{cases}$ (iii) $f(x)=\begin{cases}x, & 0<x<1\\ 2-x, & 1<x<2\\ 0, & x\ge 2\end{cases}$

3. Find Fourier Sine integral of the following functions:
(i) $f(x)=\begin{cases}x, & 0\le x\le 2\\ 0, & x>2\end{cases}$ (ii) $f(x)=\begin{cases}\sinh x, & 0\le x\le 3\\ 0, & x>3\end{cases}$ (iii) $f(x)=\begin{cases}0, & 0\le x\le 1\\ 1, & 1<x\le 2\\ 0, & x>2\end{cases}$

4. Use Fourier integral theorem to prove that $\int_0^\infty \frac{\cos\lambda x}{1+\lambda^2}\,d\lambda=\frac{\pi}{2}e^{-x}$ for all $x>0$.

5. Use Fourier integral theorem to show that $e^{-x}-e^{-2x}=\frac{6}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(1+\lambda^2)(4+\lambda^2)}\,d\lambda$ for all $x>0$.

6. Find the Fourier integral of the function $f(x)=\begin{cases}0, & x<0\\ \frac{1}{2}, & x=0\\ e^{-x}, & x>0\end{cases}$

7. Using Fourier integral theorem, show that $\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda}\sin\lambda x\,d\lambda=\begin{cases}\frac{\pi}{2}, & 0<x<\pi\\ 0, & x>\pi\end{cases}$

8. Find Fourier Sine integral of $f(x)=e^{-ax}$, $(a>0)$, and show that $\int_0^\infty \frac{\lambda\sin\lambda x}{a^2+\lambda^2}\,d\lambda=\frac{\pi}{2}e^{-ax}$ for all $x>0$.

9. Find the complex Fourier integral of the following functions:
(i) $f(x)=\begin{cases}|x|, & -\pi<x<\pi\\ 0, & \text{Otherwise}\end{cases}$ (ii) $f(x)=\begin{cases}\sinh x, & |x|<a\\ 0, & |x|\ge a\end{cases}$

10. Let $f(x)$ be a function defined on $(0,\infty)$, whose Fourier cosine integral coefficient is $A(\lambda)$, then show that at points of continuity $x^2 f(x)=\frac{2}{\pi}\int_0^\infty A^*(\lambda)\cos\lambda x\,d\lambda$, where $A^*(\lambda)=-A''(\lambda)$.
(*Note: The factor $\frac{2}{\pi}$ is absent in the above result if it is included in the coefficient.*)

Fourier Transforms:

1. Find Fourier transform of the following functions:
   (i) e^{-at^2} (ii) $e^{-a|t|}$ (iii) $e^{-at}u_0(t)$ where, $a > 0$.
2. Find the solution of the following differential equations:
   (i) $y' - 4y = H(t)e^{-4t}, -\infty < t < \infty$ (ii) $y'' + 5y' + 4y = \delta(t-2), -\infty < t < \infty$.
3. Let $\mathscr{F}[f(t)] = F(\omega)$ and $F(0) = 0$, then prove that $\mathscr{F}[\int_{-\infty}^{t} f(\tau)d\tau] = \dfrac{1}{i\omega}F(\omega)$.
4. If $\mathscr{F}[f(t)] = F(\omega)$, then prove that
   $\mathscr{F}[f(t)\sin(\omega_0 t)] = \dfrac{1}{2}\left[F(\omega+\omega_0) - F(\omega-\omega_0)\right]$,$\omega_0$ be any real number.
5. State and prove symmetry property of Fourier Transform.
6. Evaluate the following:
   (i) $\mathscr{F}\left[\dfrac{1}{5+it}\right]$ (ii) $\mathscr{F}\left[t^2 e^{-5|t|}\right]$.
7. State frequency convolution theorem and use it to prove $\int_{-\infty}^{\infty} \dfrac{d\tau}{(2-i\tau+i\omega)(2+i\tau)} = \dfrac{2\pi}{4+iw}$.
8. Find the inverse Fourier transform of following functions:
   (i) $\dfrac{e^{4i\omega}}{3+i\omega}$ (ii) $\dfrac{1}{12+7i\omega-\omega^2}$ (iii) $\dfrac{i\omega}{(i\omega+2)(i\omega+3)}$ (iv) $\omega e^{-\frac{\omega^2}{16}}$ (v) $\dfrac{1}{(i\omega+k)^2},\ k > 0$.
9. Find the Fourier Cosine and Sine transforms of the following functions:
   (i) $f(t) = e^{-t}, t \geq 0$ (ii) $f(t) = \begin{cases} \cos t, & 0 \leq t \leq a \\ 0, & t > a \end{cases}$.
10. Find Fourier transform of the function $f(x) = \begin{cases} 1 - x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$.

    Hence evaluate the integrals: (i) $\int_0^\infty \dfrac{x\cos x - \sin x}{x^3}dx$ (ii) $\int_0^\infty \dfrac{x\cos x - \sin x}{x^3}\cos\left(x/2\right)dx$.