Visvesvaraya National Institute of Technology, Nagpur
Department of Mathematics
Assignment-1 (Functions of several variables)
Limits, continuity and differentiability **Subject: MAL-102**

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1. Find the limits of the following functions as $(x,y) \to (0,0)$.

   $(i)\ \lim \dfrac{x^3y^3}{x^2+y^2}$, $(ii)\ \lim \dfrac{x^4+y^4}{x^2+y^2}$, $(iii)\ \lim \dfrac{x\sin(x^2+y^2)}{x^2+y^2}$.

2. Show that the following limits does not exist.

   $(i)\ \lim_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^4}$, $(ii)\ \lim_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x-y}$, $(iii)\ \lim_{(x,y)\to(0,1)} \tan^{-1}(\dfrac{y}{x})$

3. Evaluate the following using ε, δ definition

   $(i)\ \lim_{(x,y)\to(1,2)} (x^2+2y) = 5$ $(ii)\ \lim_{(x,y)\to(0,0)} \dfrac{x^3}{x^2+y^2} = 0$, $(iii)\ \lim_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$ does not exist.

4. Show that the function $f(x,y) = \begin{cases} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & \text{if } (x,y) \neq (0,0) \\ \frac{1}{2} & \text{if } (x,y) = (0,0) \end{cases}$, is continuous at the point $(0,0)$.

5. Discuss about the continuity of the function $f(x,y) = \begin{cases} \frac{x^4-y^2}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$, at the point $(0,0)$.

6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. Show that the function satisfies the following
   (i) The iterate limits $\lim_{x\to 0}(\lim_{y\to 0} f(x,y))$ and $\lim_{y\to 0}(\lim_{x\to 0} f(x,y))$ exist and equal to zero.
   (ii) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.
   (iii) $f(x,y)$ is not continuous at $(0,0)$.
   (iv) The partial derivatives exist at $(0,0)$.

7. Let $f(x,y)$ be defined in $S = \{(x,y) \in \mathbb{R}^2 : a < x < b, c < y < d\}$. Suppose that the partial derivatives of f exist and are bounded in S. Then show that f is continuous in S.

8. Let $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$, prove that
   (i) $f_x(0,y) = -y$ and $f_y(x,0) = x$ for all x and y,
   (ii) $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$ and
   (iii) $f(x,y)$ is differentiable at $(0,0)$.

9. Let $f(x,y) = (x^2+y^2)\sin(\frac{1}{x^2+y^2})$ if $(x,y) \neq (0,0)$ and 0, otherwise. Show that f is differentiable at every point of $\mathbb{R}^2$ but the partial derivatives are not continuous at $(0,0)$.

10. Suppose $f(x,y)$ is a function with $f_x(x,y) = f_y(x,y) = 0$ for all (x,y). Then show that $f(x,y) = c$, a constant.