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Department of Mathematics

Assignment-1 **Linear Algebra and Applications (MAL206)**

1. Prove that: if A and B are row-equivalent $m \times n$ matrices, the homogeneous systems of linear equations $AX = 0$ and $BX = 0$ have exactly the same solutions.

2. Prove that: if A is an $m \times n$ matrix and $m < n$, then the homogeneous system of linear equations $AX = 0$ has a non-trivial solution.

3. Find a row-reduced echelon matrix which is row-equivalent to $A = \begin{bmatrix} 1 & -i \\ 2 & 2 \\ i & 1+i \end{bmatrix}$. What are the solutions of $AX = 0$?

4. Obtain row-reduced echelon form for the following matrices
$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{bmatrix}.$$

5. Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}$. For which triples (y_1, y_2, y_3) does the system $AX = Y$ have a solution?

6. Define the rank of a matrix. Find the rank of the following matrices
$$\begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix}.$$

7. Define the terms: field, vector space, subspace, basis, dimension, linear combination, linear dependence, sum and direct sum.

8. True/False
   (a) Every vector space contains a zero vector.
   (b) A vector space may have more than one zero vector.
   (c) In any vector space $ax = ay$ implies that $x = y$.
   (d) A vector in F^n may be regarded as a matrix in $M_{n\times 1}(F)$.
   (e) In the polynomial space $P(t)$ over some field F, only polynomials of the same degree can be added.

9. A function $f : \mathbb{R} \to \mathbb{R}$ (reals) is an even function if $f(-x) = f(x)$ for each real number x. Prove that the set E of all even functions with the operations of addition and scalar multiplication defined by $(f + g)(x) = f(x) + g(x)$, and $kf(x) = (kf)(x)$, where $f, g \in E$; $\;k \in \mathbb{R}$ is a vector space. What about the set O of all odd functions?

10. Define addition and scalar multiplication on $V = \mathbb{R}^2$ by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + 2y_1, x_2 + 3y_2) \text{ and } c(x_1, x_2) = (cx_1, cx_2).$$

Is V a vector space over $\mathbb{R}$ with these operations?

11. Prove that a non-empty subset of a vector space is a subspace if and only if it is closed under addition and scalar multiplication.

12. Prove that the intersection of any number of subspaces of a vector space is a subspace. Is the union of subspaces of a vector space also a subspace? Justify your answer.

13. Consider the matrix space $M_{n\times n}$ over $\mathbb{R}$. Let U be subspace of upper triangular matrices and L the subspace of lower triangular matrices. Show that V is the sum of U and L but not the direct sum.

14. Let W_1 and W_2 be two subspaces of a vector space V such that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. Prove that each vector $v \in V$ there are unique vectors $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $v = \alpha_1 + \alpha_2$.

15. Consider the space $\mathbb{R}^3$. Determine whether $u_1 = (1, 2, -3)$, $u_2 = (1, -3, 2)$ and $u_3 = (2, -1, 5)$ are linearly independent?

16. Consider the matrix space $M_{2\times 2}$. Determine whether the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -5 \\ -4 & 0 \end{bmatrix}$ are linearly independent?

17. Consider the polynomial space $P_3(t)$. Determine whether the polynomials, $u = t^3+4t^2-2t+3$, $v = t^3 + 6t^2 - t + 4$ and $w = 3t^3 + 8t^2 - 8t + 7$ are linearly dependent?

18. Prove that every two bases of a finite dimensional vector space have same number of elements.

19. Show that the columns of every invertible $n \times n$ matrix give a basis for $\mathbb{R}^n$.

20. Let V be a vector space and dim $V = n$. Then any subset of V which contains more than n elements is linearly dependent.

21. Let W_1 and W_2 be finite dimensional subspaces of a vector space V. Show that $W_1 + W_2$ is finite dimensional and

$$\dim W_1 + \dim W_2 = \dim (W_1 + W_2) + \dim (W_1 \cap W_2).$$

22. Show that the subspace U of $\mathbb{R}^4$ spanned by the vectors $u_1 = (1, 2, -1, 3)$, $u_2 = (2, 4, 1, -2)$ and $u_3 = (3, 6, 3, -7)$ and the subspace W of $\mathbb{R}^4$ spanned by the vectors $v_1 = (1, 2, -4, 11)$ and $v_2 = (2, 4, -5, 14)$ are equal; that is, $U = W$.

23. Define coordinate vector of a vector relative to a basis.

24. Consider real space $\mathbb{R}^3$. Show that the set of vectors $S = \{u_1, u_2, u_3\}$, is a basis for $\mathbb{R}^3$, where $u_1 = (1, -1, 0)$, $u_2 = (1, 1, 0)$ and $u_3 = (0, 1, 1)$. Find the coordinate vector of $v = (5, 3, 4)$ relative to S.

25. Consider the polynomial space $P_2(t)$ over $\mathbb{R}$. Show that the set $S = \{1, t - 1, (t - 1)^2\}$ is a basis for $P_2(t)$. Find the coordinate vector of $v = 2t^2 - 5t + 6$ relative to S.

26. Let P be the change-of-basis matrix from a basis S to a basis S' in a vector space V. Then, for any vector $v \in V$, we have $P[v]_{S'} = [v]_S$ and hence $P^{-1}[v]_S = [v]_{S'}$.