Visvesvaraya National Institute of Technology,Nagpur
Department of Mathematics
Assignment-1

1. let $f(x) = [x]$, for all $x \in \mathbb{R}$, and let p be any integer. Find the limit of $f(x)$ as $t \to p$. ($[t]$ denotes the greatest integer $\leq t$).

2. Use the relation $\lim_{x\to 0} \frac{\sin x}{x} = 1$ to establish the following limit formulas

$$(i) \lim_{x\to 0} \frac{\tan 2x}{\sin x} = 2 \qquad (ii) \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.$$

3. A function f is defined as follows:

$$(i) f(x) = \begin{cases} \sin x, & \text{if } x \leq c, \\ ax+b, & \text{if } x > c, \end{cases} \qquad (ii) f(x) = \begin{cases} 2\cos x, & \text{if } x \leq c, \\ ax^2+b, & \text{if } x > c, \end{cases}$$

where a, b, c are constants. If b and c are given, find all values of a(if any exist) for which f is continuous at the point $x = c$.

4. Using the ϵ, δ definition of continuity,show that $\sin x$ is continuous at $x = 0$. (or) Show that $|\sin x| \leq |x|, \forall x \in \mathbb{R}$.

5. $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous at every point $c \in \mathbb{R}$.

6. Show that the absolute value function $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$ but not differentiable at $x = 0$.

7. Show that $f(x) = x^{\frac{1}{3}}, x \in \mathbb{R}$ is not differentiable at $x = 0$.

8. Assume that f is differentiable at a. Find

$$\lim_{x\to a} \frac{xf(a) - af(x)}{x-a}.$$

9. Show that the function $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$ is differentiable at all $x \in \mathbb{R}$. Also show that the function $f'(x)$ is not continuous at $x = 0$. Thus, a function that is differentiable at every point of $\mathbb{R}$ need not have a continuous derivative $f'(x)$.

10. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function (i.e,$f(-x) = f(x)$ for all $x \in \mathbb{R}$) and has a derivative at every point, then the derivative f' is an odd function (i.e.,$f(-x) = -f(x)$ for all $x \in \mathbb{R}$).

11. Let $f(0) = 0$ and $f'(0) = 1$. For positive integer k, show that

$$\lim_{x\to 0} \frac{1}{x} \left\{ f(x) + f(\frac{x}{2}) + f(\frac{x}{3}) + ... + f(\frac{x}{k}) \right\} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k}.$$

12. Let $f(x) = 1 - x^{\frac{2}{3}}$. Show that $f(1) = 0 = f(-1)$, but $f'(x)$ is never zero in the interval [-1,1]. Explain how this possible in view of Rolle's theorem.

13. Assume that $a_0, a_1, ...a_n$ are real numbers such that

$$\frac{a_0}{n+1} + \frac{a_1}{n} + ... + \frac{a_{n-1}}{2} + a_n = 0.$$

Prove that $p(x) = a_0 x^n + a_1 x^{n-1} + ... + a_n$ has at least one root in (0,1).

14. Let $f : [a, b] \to \mathbb{R}$ be differentiable. If $f'(x) = 0$ for all $x \in [a, b]$, show that f is constant on $[a, b]$.

15. Show that the equation $f''(x) = 0$ has atleast one real root between a and b if f, f', f'' are continuous in $a \leq x \leq b$ and the curve $y = f(x)$ crosses $x-$axis atleast three distinct points between a and b inclusive.

16. Show that between any two roots of $e^x \cos x - 1 = 0$, there exists atleast one root of $e^x \sin x - 1 = 0$.

17. Using Lagrange Mean Value theorem, show that
(i) $x < -\log(1-x) < \frac{x}{1-x}$, for $x \in (0, 1)$.
(ii) $e^x \geq 1 + x$, for $x \in \mathbb{R}$

18. Verify the Lagrange Mean Value theorem for the function $\sin^{-1} x$ in $(0, 1)$.

19. Using Cauchy Mean Value theorem, show that
(i) $1 - \frac{x^2}{2!} < \cos x$, for $x \neq 0$.
(ii) $x - \frac{x^3}{3!} < \sin x$, for $x > 0$.
(iii) $\cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, for $x \neq 0$.
(iv) $\sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$, for $x > 0$.

20. Show that $\sqrt{\frac{1-x}{1+x}} < \frac{\ln(1+x)}{\sin^{-1} x} < 1$, for $0 \leq x < 1$.

21. Verify Taylor's theorem for $f(x) = (1-x)^{\frac{5}{2}}$ with Lagrange form of remainder upto 2 terms in the interval $[0, 1]$.

22. Show that, for all $x \geq 0$
$1 + x + \frac{x^2}{2} \leq e^x \leq 1 + x + \frac{x^2}{2} e^x$.

23. Show that for all x,
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^n - 1 \frac{x^2 n-1}{(2n-1)!} + (-1)^n \frac{x^2 n}{(2n)!} \sin \theta x$, where $0 < \theta < 1$.