Visvesvaraya National Institute of Technology, Nagpur

Department of Mathematics

Assignment-5: Sequence and Infinite Series **Course: MAL-101**

Sequences:

1. Discuss the convergence or otherwise of the following sequences?
   (i) $\left\{(-1)^{n+1}\frac{1}{n}\right\}$ (ii)$\left\{\frac{n}{5^n}\right\}$ (iii) $\left\{\frac{3n+1}{n+1}\right\}$ (iv) $\left\{\frac{(2n+3)!}{(n+1)!}\right\}$ (v)$\left\{\frac{2^n3^n}{n!}\right\}$ (vi)$\left\{\frac{1}{\sqrt{n}}\right\}$
   (vii)$\{a_n\}$, where $a_n = \frac{1}{1+n^2} + \frac{2}{2+n^2} + ... + \frac{n}{n+n^2}$.
   (viii)$\{a_n\}$, where $a_n = 1 + \frac{1}{2!} + ... + \frac{1}{n!}$. (ix)$\{a_n\}$, where $a_n = 1 + \frac{1}{2} + ... + \frac{1}{n}$.

2. Find the limits of the following sequences:
   a)$\left\{\frac{\sin(n)}{n}\right\}$ b)$\left\{\sqrt{\frac{n+1}{n}}\right\}$ c) $\{\sqrt[n]{p}\}$, where $p > 0$.

3. Show that the sequence $\{r^n\}$ is
   i) divergent to $+\infty$ if $r > 1$ ii) convergent to 1 if $r = 1$
   iii) convergent to 0 if $-1 < r < 1$ iv) oscillating finitely if $r = -1$
   v)oscillating infinitely if $r < -1$

4. Is it true that a sequence $\{a_n\}$ of positive numbers must converge if it is bounded above? Give reasons for your answer.

5. Prove that the sequence $\{a_n\}$ defined by $a_1 = \sqrt{7}, a_{n+1|} = \sqrt{7 + a_n}$ for all n, is convergent and the limit of the sequence is the positive root of the equation $x^2 - x - 7 = 0$.

Infinite Series:

6. Find a_n and the sum of the series $\sum_{n=1}^{\infty} a_n$, whose n^{th} partial sum given as
   a)$s_n = \frac{n-1}{n+1}$, b) $s_n = 3 - n2^{-n}$.

7. What is the value of c if $\sum_{n=2}^{\infty}(1+c)^{-n} = 2$?

8. Examine the behaviour of the following seies using basic principle(sequence of partial sums):
   i) $\sum_{n=1}^{\infty}\frac{1}{n}$, ii)$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$, iii)$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$, iv) $\sum_{n=1}^{\infty}5^n$, v) $\sum_{n=1}^{\infty}\frac{1}{n!}$, vi) $\sum_{n=1}^{\infty}\frac{1}{7^{n-1}}$

9. Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms, is convergent then show that

(i)$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$, is convergent ii) $\sum_{n=1}^{\infty} a_n^2$ is convergent iii) $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is divergent.

10. Test for the convergence of the following series $\sum_{n=1}^{\infty} a_n$:

i)$a_1 = 2, a_{n+1} = \left(\frac{1+\sin n}{n}\right) a_n$ for all n. ii)$a_1 = 1, a_{n+1} = \left(\frac{1+\arctan n}{n}\right) a_n$ for all n.

iii) $a_1 = 3, a_{n+1} = \left(\frac{n}{n+1}\right) a_n$ for all n. iv) $a_1 = 5, a_{n+1} = \left(\frac{n^{\frac{1}{n}}}{2}\right) a_n$ for all n.

v)$a_1 = \frac{1}{3}, a_{n+1} = \sqrt[n]{a_n}$ for all n vi)$a_1 = \frac{1}{2}, a_{n+1} = (a_n)^{n+1}$ for all n

11. Examine the behaviour of the following series. Also find their sum, if exists?

i) $2+\frac{2}{3}+\frac{2}{9}+....+\frac{2}{3^{n-1}}+...$ $ii)1-2+4-8+....+(-1)^{n-1}2^{n-1}+....$ iii) $\frac{5}{2.3}+\frac{5}{3.4}+....+\frac{5}{(n+1)(n+2)}+....$

iv) $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}-\frac{(-1)^n}{5^n}\right)$ v) $\sum_{n=1}^{\infty}\left(1-\frac{3}{n}\right)^n$ vi) $\sum_{n=1}^{\infty} n!e^{-n}$

12. Examine the behaviour of the following series(converges or diverges):

i)$\sum_{n=1}^{\infty} \tan\frac{1}{n}$ ii) $\sum_{n=1}^{\infty} \frac{\coth n}{n^2}$ iii) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ iv)$\sum_{n=1}^{\infty} \frac{(\ln)^n}{n^n}$ v) $\sum_{n=1}^{\infty} \frac{n!\ln n}{n(n+2)!}$ vi) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+n!}$

vii) $\sum_{n=1}^{\infty} \frac{2^n}{3\sqrt{5^n+n^5}}$ viii) $\sum_{n=1}^{\infty} \frac{1}{(n!)^n}$ ix) $\sum_{n=1}^{\infty} \frac{1}{\ln\ln n}$ x) $\sum_{n=1}^{\infty}(-5)^n$ xi) $\sum_{n=1}^{\infty}(\sqrt{n^2+1}-n)x^{2n}$

xii) $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}x^n, x > o$ xiii) $\sum_{n=1}^{\infty}(\sqrt{n^4+1}-\sqrt{n^4-1})$

13. Find the radius and the interval of the convergence of the following power series:

i) $\sum_{n=0}^{\infty} \frac{(-3)x^n}{\sqrt{n+1}}$ ii) $\sum_{n=1}^{\infty} \frac{n(x-a)^n}{b^n}, b > 0$ iii) $\sum_{n=0}^{\infty} \frac{x^n}{(\ln n)^n}$ iv) $\sum_{n=1}^{\infty}(-1)^n n4^n x^n$ v)$\sum_{n=1}^{\infty} n!(2x-1)^n$

vi) $\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!}x^n$ where $k \in \mathbb{N}$. vii) $\sum \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ viii) $\sum \frac{(-1)^n x^{2n+1}}{2^{2n+1}n!(n+1)!}$

14. Find the radius and the circle of the convergence of the following power series:

i)$\sum_{n=0}^{\infty}\left(1+\frac{2}{n}\right)^{n^2} z^n$ ii) $\sum_{n=1}^{\infty}(n+a^n)z^n$ iii) $\sum_{n=0}^{\infty} \frac{1}{(n!)}\left(\frac{iz-1}{2+i}\right)^{n^2}$ iv) $\sum_{n=0}^{\infty} \frac{(1+i)^n}{(n+2)}(z+3i)^n$ v) $\sum_{n=0}^{\infty} z^{2^n}$

15. Discuss the convergence or divergence of the following improper integral.Find its value if it exists.

i) $\int_1^3 \frac{\sqrt{x}}{\ln x}\,dx$ ii)$\int_0^{\infty} \frac{x^{p-1}}{1+x}\,dx, 0 < p < 1$ iii) $\int_a^b \frac{1}{(x-a)^p}\,dx, p > 0$ iv) $\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$

v) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x\sqrt{x}}\,dx$ vi) $\int_0^3 \frac{1}{x^2-3x+6}\,dx$ vii) $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$ viii) $\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{x^m}\,dx$